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Quantization of Fractal Systems: One-Particle Excitation States

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We consider a self-similar chain of quantum harmonic oscillators as a model of quantum field theory on a fractal supporter. In terms of this model a mass generation mechanism for one-particle excitations is proposed.

1. INTRODUCTION

The common way in which the concept of energy is introduced into quantum mechanics is twofold. On the one hand, the energy of a particle (or a system) is a quantity conserved due to the invariance with respect to time translations (i.e., the eigenvalue of the $\partial/\partial t$ generator). On the other hand we have to use the mechanical analogy $E = p^2/(2m)$ with p substituted by the operator $\hat{p} = -i\hbar\ \partial/\partial x$. Having done this, we can construct a field-theoretic model with the usual renormalization problems, etc.

In the present paper we propose an alternative approach which in our opinion would allow us to introduce the concept of energy in a more consistent way from a cosmological standpoint. Our approach is based on two ideas.

First, we consider self-similarity (or roughly, conformal invariance) as a fundamental property of the whole of nature, rather than a purely theoretical tool which appears in quantum field theory to cut the divergencies off.

Second, in terms of our model the (dynamical) mass of the particle itself comes from the amplitude for this particle to flip from one cell of space-time to the neighboring one.

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2. THE DIMENSION AND FRACTAL STRUCTURE OF SPACE-TIME

There are two different ways to define the physical space-time dimension. First, the dimension of space-time can be defined as the number of coordinates (numbers) which we need to define the state of a system. The second, and this is more convenient for mathematicians than physicists, is the Hausdorff–Besicovitch dimension. Here we recall the latter definition (Mandelbrot, 1988; Feder, 1988).

Let us consider a set $\mathcal{T}$ embedded into a D-dimensional Euclidean space $\mathcal{R}^D$; then we can construct a covering of the set $\mathcal{T}$ with d-dimensional balls of radius δ and define a measure function

$$M_d = \sum_{\mathcal{T}} \gamma(d)\delta^d = \gamma(d)N(\delta)\delta^d \tag{1}$$

where $\gamma(d) = \Gamma(\frac{1}{2})^d/\Gamma(1 + d/2)$ is a geometrical factor.

The set $\mathcal{T}$ is said to have Hausdorff–Besicovitch dimension D if

$$\lim_{\delta \to 0} M_d = \begin{cases} 0 & d > D \\ \infty & d < D \end{cases}$$

This definition is also referred to as a "mass dimension" since if we calculate a mass contained in a spherical volume of radius r we in fact perform a δ-covering of this volume. If we suppose the mass distribution to have dimension D, then the mass interior to a spherical volume should be proportional to r^D (say, $M = \frac{4}{3}\pi r^3$ for $D = 3$). Experimentally observed mass distributions are usually far from that. On large scales, up to 50 Mpc, we have $M(r) \sim r^{1.1}$ (Sancisi and van Albada, 1987). For smaller ones comparable to the size of nuclei we formally can estimate the dimension by studying the dependence of the radius of the nuclei on their masses. For example, using the data on α-particle elastic scattering on nuclei (Yushkov, 1993), for atomic numbers $A > 30$ we obtain approximately

$$A \sim r^{1.57} \tag{2}$$

which is also far from the $D = 3$ case.[3]

If we go down to subnuclear and even to the Planck scale, we cannot afford ourselves the luxury of experimental study, say, for $E \sim 10^{15}$ GeV, but theoretical investigations (Crane and Smolin, 1986; Altaisky *et al.*, n.d.) lead to the conclusion that the physical space-time at the scales of quantum gravity should have a dimension less than that of the embedding space, at

[3] The well-known relation $r = r_0 A^{-1/3}$ is suitable more or less strictly only for the mirror nuclei with $A < 40$.

least for the sake of anomaly cancellation. In addition, if we endow the space with fractal geometry, the behavior of Feynman loop integrals turns out to be much softer than in $D = 4$ Minkowski space.

Thus, the only region where physical space-time has been strongly proven to have 1 + 3 dimensions is that of terrestrial scales (10^{-6}–10^{9} cm).

In the present paper we discuss a possible mechanism for the origin of the (dynamical) particle mass due to a self-similar structure of space-time. Let us suppose that the universe (we shall denote it $\mathcal{U}$ in our one-dimensional toy model) is closed and has a finite mass M_0 equal to the mass of the initial singularity. A topology of self-similar sets can be introduced on $\mathcal{U}$ in a way shown in Altaisky *et al.* (n.d.) if we split it into two hemispheres, say A and B, and then perform the partitioning of both sets into fractal gaskets (see Appendix).

If we require total symmetry under $A \leftrightarrow B$, then the equality

$$E_A = E_B = \frac{M_0 c^2}{2} \tag{3}$$

must hold. This equation can be regarded as the definition of the energy, with c^2 being a formal proportionality coefficient (no relativistic concepts have been used in our model). According to our model the sizes of the sets A and B also should be equal:

$$l_A^0 = l_B^0 \equiv l_0 = \frac{L_0}{2} \tag{4}$$

Implying the color reverse procedure described in the Appendix, in the ith generation we obtain

$$l_i = l_0 3^{-i}, \qquad m_i = m_0 3^{-i} \tag{5}$$

Up to this point we have dealt with pure geometry without any dynamical degrees of freedom. Now we introduce the physical excitation states of the model. Let us use a formal mechanical analogy and suppose that in the course of the sequential partitioning process energy conservation is provided as follows. Each central part of the black domain (see Fig. 1) of the $(i - 1)$th generation is divided into three equal parts with the energy of the central part in the form of the energy of a contracted spring:

$$m_{i-1}c^2 = 2m_i c^2 + \frac{k_i}{2} (\Delta l_i)^2 \tag{6}$$

with $\Delta l_i = l_i$, and hence

$$m_i c^2 = \frac{k_i}{2} l_i^2, \qquad k_i = k_0 3^i, \qquad \text{where} \quad k_0 = \frac{2m_0 c^2}{l_0^2} \tag{7}$$

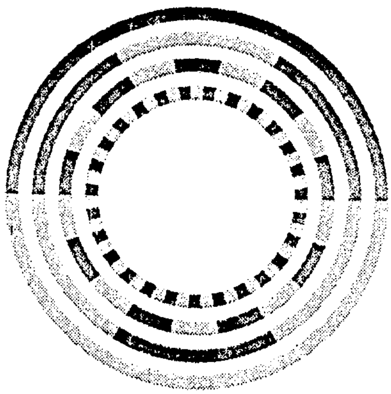

Fig. 1. Since the sphere S^n, as a boundary of the $(n + 1)$-dimensional simplex, can be partitioned into a set of $n + 1$ simplices (the idea that this partitioning could be of physical interest comes from Regge), it can also be covered with fractal gaskets, since each of which is a subset of a certain simplex.

On the ith level we have a periodic chain which consists of 3^i oscillators of typical size $b_i = 2l_i$. An arbitrary point of $\mathcal{U}$ has, therefore, a nested system of vicinities belonging to the system of subspaces

$$V_i \subset V_{i+1} \subset V_{i+2} \subset \cdots \tag{8}$$

where $V_0 \in \{A, B\}$ is the most coarse-grained one. This is just like the case of a certain point of physical space-time found inside a certain quark, inside a certain nucleon, inside a particular nucleus, atom, galaxy, and so on. If the scale invariance is broken at a certain scale, we need a finite number of numbers to describe the location of the point, or an infinite number otherwise. The particular level i at which we study the physics is of our free choice; the whole system $\cup_i V_i = \mathcal{U}$ is the property of the universe.

3. THE DYNAMICAL ORIGIN OF MASS IN A SELF-SIMILAR PERIODIC CHAIN

Using the Hamiltonian

$$H_i = m_i \sum_{k=1}^{3^i} \frac{\dot{\phi}_k^2}{2} + k_i \sum_{\mathrm{mod}\ 3^i} \frac{(\phi_k - \phi_{k+1})^2}{2} \tag{9}$$

of the ith-level chain,[4] we suppose for the sake of simplicity that each oscillator can be in two states only, namely $|0\rangle$, with energy already incorporated into

[4] We use k to denote a particular cell of the chain (oscillator) and i to denote the generation; the latter index is omitted if the case is clear enough.

the vacuum energy of the whole system, and $|1\rangle$, the one-particle excitation state. We also suppose that there is a nonzero probability for transition of the excitation from one cell to the neighboring one,

$$\lambda_i = \langle k | k + 1 \rangle_i$$

Let $\psi(t)$ be the wave function of the excitation; then the probability amplitude to find it in a particular kth cell of a chain is

$$C_k(t) = \langle k | \psi(t) \rangle$$

The evolution equation for C_k is

$$i\hbar \frac{dC_k(t)}{dt} = \sum_l H_{kl} C_l(t) \tag{10}$$

Since we allow neighboring transitions only, we have two equal nonzero amplitudes,

$$A = \frac{k_i}{2} \langle k | (\phi_k - \phi_{k+1})^2 | k + 1 \rangle \tag{11}$$

which determine the transition probability. The stationary state energy is equal to

$$E_0 = \langle k | H | k \rangle = \hbar\omega_i \tag{12}$$

since we incorporate the zero mode into the vacuum energy

$$\frac{\hbar\omega_i}{2} = \frac{k_i(\Delta l_i)^2}{2}. \tag{13}$$

Explicitly substituting

$$\phi^2_{(i)} = \frac{\hbar}{m_i\omega_i} Q^2_{(i)}, \qquad \text{where} \quad Q = \frac{a + a^*}{\sqrt{2}}$$

into (11), we get

$$A_{(i)} = \sqrt{2} \frac{c\hbar}{l_0} \lambda_i 3^i \tag{14}$$

Bearing in mind the continuous limit of the chain ($i \to \infty$, $b \to 0$), we rewrite the evolution equation in the form (Feynman *et al.*, 1965; Sidharth, 1994)

$$i\hbar \frac{\partial C(x_k, t)}{\partial t} = E_0 C(x_k, t) - AC(x_k + b, t) - AC(x_k - b, t) \tag{15}$$

where x_k stands for the number of the particular cell. In the continuous limit of (15) we obtain the Schrödinger-type equation

$$i\hbar \frac{\partial C(x)}{\partial t} = -\frac{\hbar^2}{2m_{\text{eff}}} \frac{\partial^2 C(x)}{\partial x^2} \tag{16}$$

where

$$m_{\text{eff}} = \frac{\hbar^2}{2Ab^2}$$

is the effective mass of the excitation [see Feynman *et al.* (1965) for details].

For the ith generation of our model ($b_i = 2l_i$) the effective mass of excitation turns out to be

$$m_{\text{eff}}^{(i)} = \frac{\hbar 3^i}{8(2cl_0\lambda_i)^{1/2}} \tag{17}$$

It is worth noting that equation (13), which has been used to set $E_0 = \hbar\omega_i$, is not the unique solution of the problem. In principle, there is the possibility of choosing

$$\frac{\hbar_{(i)}\omega_i}{2} = \frac{k_i(\delta l_i)^2}{2} \tag{18}$$

with

$$\hbar_{(i)} = \sqrt{2}m_0 l_0 c 3^{-2i} \tag{19}$$

treated as a scale-dependent quantity. In this case

$$m_{\text{eff}}^{(i)} = \frac{m_0 3^{-i}}{8\lambda_i} \tag{20}$$

In terms of our one-dimensional model we cannot choose in a credible way which of the possibilities is better. We even cannot strictly identify our $\hbar$ with the physical Planck constant. We can only mention that if $\hbar_{\text{phys}}$ is a constant determining the breaking of scale invariance at the l_{break} level of the subdivision process, then

$$\hbar = 2^{1/2} m_0 l_0 c 3^{-2l_{\text{break}}} \tag{21}$$

Thus, if we fix the scale at which the symmetry breaking takes place, then $\hbar$ and c turn out to be no longer independent.

4. CONCLUSION

In conclusion we mention that the relation between scale invariance and the zero-mode (or vacuum) energy goes far beyond our simple model.

If we accept the hypothesis that scale invariance is a fundamental property of the whole universe, then the first thing to do is to use the representation of the scale transformation group to construct quantum field theory models. Doing so, we should write the state vectors in Hilbert space in the form

$$|f\rangle = \frac{1}{\sqrt{c_g}} \int_G |g; y\rangle \, d\mu(y) \, \langle g; y | f \rangle \tag{22}$$

The difference from the simple case of Fourier decomposition is that in the case of an arbitrary group G we need to choose a certain vector $g \in H$ of our Hilbert space such that the normalization constant

$$c_g = \|g\|^{-2} \int_G d\mu(y) \, |\langle g, U(y)g \rangle|^2 < \infty \tag{23}$$

is finite. Here $y = \{y_i\}$ are coordinates on G, $|g; y\rangle \equiv U(y)g$ is a cyclic representation of G, and $d\mu(y)$ is a left-invariant measure on G. In the one-dimensional case, which is sufficiently instructive, we can locally perform the decomposition with respect to the coherent states representation of the affine group (Daubechies *et al.*, 1986):

$$G: \quad x \to ax + b, \qquad d\mu = \frac{da\, db}{a^2} \tag{24}$$

$$g_n(x) = (-1)^n \frac{\pi^{-1/4}}{2^n n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2} \tag{25}$$

$$\langle x | f \rangle = \frac{1}{\sqrt{c_g}} \int \frac{1}{\sqrt{a}} g\left(\frac{x-b}{a}\right) \langle g; a, b | f \rangle \frac{da\, db}{a^2} \tag{26}$$

But, alas, in contrast to common quantum mechanics, the ground state $g_0(x)$ does not fit the admissibility condition (23), and that is the source of the whole problem (Altaisky, 1994).

APPENDIX
Fractal Gaskets and Sphere Covering

Let us recall the construction procedure of the Sierpinski hypergasket (see e.g., Eyink, 1989a,b).

Partitioning the unit d-simplex in R^d into $(d + 2)$ subsimplices of edge length 1/2, one (a) removes the open central subsimplex, and (b) repeats the operation with the $(d + 1)$ closed subsimplices.

Sierpinski gaskets obtained in this way can be used for triangulation of an S^n-sphere. Their self-similarity is very relevant to RG applications. Their shortcomings are also evident. They are not invariant under translations, even inside a single gasket, and they are not dense in the embedding space.

Let us modify the gasket-generating procedure. To clarify the consideration, let us imagine a unit simplex of black color. In the first step of the recursive procedure we remove the central open part of it; the central subsimplex becomes white, and then—here is the difference—we repeat the procedure with *all* $(d + 2)$ subsimplices. The generalization to white pieces seems evident: the central part of each simplex reverses its color.

Since the numbers of black and white subsimplices at the $(k + 1)$ stage of the recursive procedure are given by

$$n_W^{k+1} = (d + 1)n_W^k + n_B^k \tag{A1}$$

$$n_B^{k+1} = (d + 1)n_B^k + n_W^k$$

for asymptotically large k we obtain

$$n_k \approx \frac{1}{2}(d + 2)^k$$

simplices of each color of $\delta = 2^{-k}$ edge size. The fractal dimension of the constructed set is

$$D = \frac{\log(d + 2)}{\log 2} \tag{A2}$$

rather than

$$D = \frac{\log(d + 1)}{\log 2} \tag{A3}$$

for the Sierpinski gasket.

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